October Math Gems

Problem of the week 4

§1 problems

Problem 1.1. In the figure below ABC is a right-angled triangle and BD an angle bisector. If $AB = 3$, and the area of $ABD = 9$, what is the length of DC?

te.jpeg

Proof. In $\triangle ADB$,

$$\frac{3}{\sin\angle ADB} = \frac{AD}{\sin\angle ABD} = \frac{BD}{\sin\angle A} \implies BD = \frac{AD \times \sin\angle A}{\sin\angle ABD}$$

In $\triangle BDC$,

$$\frac{DC}{\sin\angle DBC} = DB = \frac{BC}{\sin\angle CDB}$$

$$\text{Area of } \triangle ABD = \frac{1}{2} \times 3 \times AD \times \sin\angle A = 9 \implies AD \times \sin\angle A = 6$$

$$\frac{DC}{\sin\angle DBC} = \frac{AD \times \sin\angle A}{\sin\angle ABD} = \frac{6}{\sin\angle ABD} \implies DC = 6$$

$$[\angle ABD = \angle DBC]$$

□

Problem 1.2. If a, b, c are positive reals with $abc = 1$. what is the minimum value of

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}$$

?

answer. multiply the following by $(abc)^2$ we get,

$$\frac{b^2c^2}{ab+ac}+\frac{a^2c^2}{ba+bc}+\frac{a^2b^2}{ac+cb}$$

it will not change of it as $abc = 1$.

$$\frac{b^2c^2}{ab+ac}+\frac{a^2c^2}{ba+bc}+\frac{a^2b^2}{ac+cb} \geq \frac{(bc+ac+ab)^2}{2(ac+ab+cb)} = \frac{(bc+ac+ab)}{2}$$

$$\frac{(bc+ac+ab)}{2} \geq \frac{3\sqrt[3]{(abc)^2}}{2} = \frac{3}{2}$$

So the minimum value of the expression is $\frac{3}{2}$. □

Problem 1.3. How many ordered pairs of positive integers (M, N) satisfy the equation $\frac{M}{6} = \frac{6}{N}$?

answer.

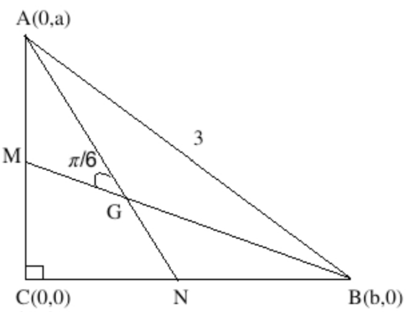
$$MN = 36$$

Factors of 36: $\{1, 36, 4, 9, 3, 12, 6, 2, 18\}$
ordered pairs of (M, N)

$$\{(1, 36), (6, 6), (36, 1), (4, 9), (9, 4), (12, 3), (3, 12), (2, 18), (18, 2)\}$$

There are 9 ordered pairs. □

Problem 1.4. Two median are drawn from acute angles a right angled triangle intersect at an angle $\frac{\pi}{6}$. If the length of the hypotenuse of the triangle = 3, then, find the area of triangle=



answer.

$$G = (\frac{b}{3}, \frac{a}{3})$$

slope of $AG = \frac{-2a}{b}$
slope of $MG = \frac{-a}{2b}$

$$\tan 30 = \frac{\frac{-2a}{b} - \frac{-a}{2b}}{1 + \frac{a^2}{b^2}}$$

$$\frac{1}{\sqrt{3}} = \frac{3ab}{2a^2 + b^2}$$

$$\frac{1}{2}ab = \frac{a^2+b^2}{3\sqrt{3}}$$

In a right angled triangle

$$AC^2 + BC^2 = AB^2 = 9$$

Then

$$a^2 + b^2 = 9$$

area=

$$\frac{1}{2}ab = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

□

Problem 1.5. If the points $a(3,4), b(7,12), andp(x,y)$ are such that $(pa^2 > (pb)^2 > (ab^2))$ Evaluate x where x is integral number.

answer. Consider the first condition

$$(x-3)^2 + (x-4)^2 > (x-7)^2 + (x-12)^2$$

$$2x^2 - 14x + 25 > 2x^2 - 38x + 193 ==> 24x - 168 > 0 ==> x > 7$$

Now, consider the second condition

$$(x-7)^2 + (x-12)^2 > (3-7)^2 + (4-12)^2 = 80$$

All that satisfied when $x = 16$ or $x = 20$ □

Problem 1.6. Known that $a + b + c = \pi$, then

$$\frac{\sin 2a + \sin 2b + \sin 2c}{\cos a + \cos b + \cos c - 1} =$$

answer. We have,

$$\sin 2a + \sin 2b + \sin 2c = 2\sin(\frac{2a+2b}{2})\cos(\frac{2a-2b}{2}) + \sin 2c =$$

$$2\sin(a+b)\cos(a-b) + 2\sin c\cos c$$

from $a + b + c = \pi$ we get

$$2\sin c\cos(a-b) + 2\sin c\cos c = 2\sin c(\cos(a-b) - \cos(a+b))$$

$$= 2\sin c(2\sin a\sin b) = 4\sin a\sin b\sin c \qquad (i)$$

$$\cos a + \cos b + -1 = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2}) + (1 - 2\sin^2\frac{c}{2}) - 1$$

Noticing that $\cos(\frac{a+b}{2}) = \cos(\frac{\pi}{2} - \frac{c}{2}) = \sin\frac{c}{2}$

$$2\sin\frac{c}{2}\cos(\frac{a-b}{2}) - 2\sin^2\frac{c}{2} = 2\sin\frac{c}{2}\cos(\frac{a-b}{2}) - 2\sin\frac{c}{2}\cos(\frac{a+b}{2}) =$$

$$2\sin\frac{c}{2}(\cos(\frac{a-b}{2}) - \cos(\frac{a+b}{2})) = 4\sin\frac{c}{2}\sin\frac{a}{2}\sin\frac{b}{2} \qquad (ii)$$

from (i) and (ii) we get

$$\frac{4\sin a\sin b\sin c}{4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}$$

$$\because \frac{\sin\theta}{\sin\frac{\theta}{2}} = 2\cos\frac{\theta}{2}$$

$$\frac{4\sin a\sin b\sin c}{4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}} = 8\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}$$

□

Problem 1.7. Given $g(x) = 9\log_8(x-3) - 5$, $g^{-1}(13) =$

answer.

$$y = 9\log_8(x-3) - 5, y+5 = 9\log_8(x-3)\frac{y+5}{9} =$$

$$\log_8(x-3), x-3 = 8^{\frac{y+5}{9}} x = 8^{\frac{y+5}{9}} + 3$$

Now, substitute x with y to get

$$g^{-1}(x) = 8^{\frac{x+5}{9}} + 3$$

Hence,

$$g^{-1}(13) = 8^{\frac{13+5}{9}} + 3 = 67$$

□

Problem 1.8. let $x, y, z > 0$. Prove that

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geq \frac{9}{x+y+z}$$

answer. We'll rewrite the left-hand side a little bit as following:

$$\frac{(\sqrt{2})^2}{x+y} + \frac{(\sqrt{2})^2}{y+z} + \frac{(\sqrt{2})^2}{z+x}$$

and we'll use the following lemma:

If a, b, x, y are real numbers and $x, y > 0$, then the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$$

We deduce that:

$$\frac{(\sqrt{2})^2}{x+y} + \frac{(\sqrt{2})^2}{y+z} + \frac{(\sqrt{2})^2}{z+x} \geq \frac{(3\sqrt{2})^2}{2(x+y+z)} = \frac{9}{x+y+z}$$

□

Problem 1.9. Determine the domain of the function

$$g(x) = \cot^{-1}\left(\frac{x}{\sqrt{x^2 - \lfloor x^2 \rfloor}}\right)$$

answer. For $g(x)$ to be defined,

$$x^2 - \lfloor x^2 \rfloor > 0$$

Thus, x^2 cannot be integer, also 0 is restricted. □

Problem 1.10. Let α, β, and γ denote the angles of a triangle. Show that

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$
$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\sin\beta\sin\gamma$$
$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=-4\sin 2\alpha\sin 2\beta\sin 2\gamma.$$

Proof. Call the three identities (a), (b), and (c), respectively. If

$$\alpha+\beta+\gamma=\pi,$$

then

$$(\pi-2\alpha)+(\pi-2\beta)+(\pi-2\gamma)=\pi.$$

We can pass from (a) to (b), and also from (b) to (c), by substituting $\pi-2\alpha, \pi-2\beta$, and $\pi-2\gamma$ for α, β, and γ, respectively. It remains to verify (a), which can be done in many ways. For instance, substitute $2u, 2v$, and $\pi-2u-2v$ for α, β, and γ, respectively. Then (a) becomes

$$\begin{gathered}\sin u\cos u+\sin v\cos v\\=[2\cos u\cos v-\cos(u+v)]\sin(u+v).\end{gathered}$$

Use the addition theorems of cosine and sine. □

Problem 1.11. Prove that no number in the sequence

$$11, 111, 1111, 11111, \ldots.$$

is the square of an integer.

answer. If s is a number in the sequence, s must have the form

$$11+100m=4(25m+2)+3,$$

where m is a non-negative integer, and therefore s leaves a remainder of 3 when divided by 4 . But squares are of the form $4n^2$ or $4n^2+4n+1$ and hence leave remainders of either 0 or 1 when divided by 4 . □

Problem 1.12. Solve the following system of three equations for the unknowns x, y and z :

$$\begin{aligned}5732x+2134y+2134z&=7866\\2134x+5732y+2134z&=670\\2134x+2134y+5732z&=11464\end{aligned}$$

Proof. The simplest expression that is symmetric in x, y, and z is their sum. Adding the three proposed equations, we obtain

$$\begin{gathered}10000x+10000y+10000z=20000\\x+y+z=2\end{gathered}$$

By subtracting

$$2134x+2134y+2134z=4268$$

from each of the three proposed equations, we obtain three new equations that when solved yield $x=1, y=-1, z=2$, respectively. □

Problem 1.13. A pyramid is called "regular" if its base is a regular polygon and the foot of its altitude is the center of its base. A regular pyramid has a hexagonal base the area of which is one quarter of the total surface-area S of the pyramid. The altitude of the pyramid is h. Express S in terms of h.

answer. Pass a plane through the altitude of the pyramid and through the midpoint of one side (of length a) of its base. The intersection of this plane with the pyramid is an isosceles triangle that can be used as a key figure: its height is h, its legs are of length l (where l is the height of a lateral face of the pyramid), and its base is $2b$ (where b is the altitude of one of the six congruent equilateral triangles composing the base of the pyramid). The area of the base is

$$\frac{S}{4} = \frac{6ab}{2},$$

the area of the lateral surface is

$$\frac{3S}{4} = \frac{6al}{2},$$

and so

$$l = 3b.$$

Using the key figure, we obtain and so

$$h^2 + b^2 = l^2 = 9b^2$$

$$b^2 = \frac{h^2}{8}$$

We also have and so

$$b^2 + \frac{a^2}{4} = a^2$$

Therefore

$$a^2 = \frac{4b^2}{3} = \frac{h^2}{6}$$

$$S = 12ab = h^2\sqrt{3}.$$

□

Problem 1.14. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b}+2\right) + \frac{b}{a}\left(\frac{b}{a}+2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}$$

answer. Adding 3 to both sides of the given equation yields

$$\left(\frac{a}{b}+\frac{b}{a}+1\right)^2 = 2025,$$

which implies that $\frac{a}{b}+\frac{b}{a}+1 = 45$. Then $\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2 = \frac{a}{b}+2+\frac{b}{a} = 46$. In particular, the original equation is satisfied by $a = 22+\sqrt{483}$ and $b = 1$. □

Problem 1.15. Solve

$$x^{x^{x^{2021}}} = 2021$$

answer. We will find if we put $x^{2021} = 2021$, So

$$x^{2021} = 2021 \implies x = 2021^{\frac{1}{2021}}$$

□

Problem 1.16. If α, β, γ do not differ by a multiple of π and if

$$\frac{\cos(\alpha+\theta)}{\sin(\beta+\gamma)} = \frac{\cos(\beta+\theta)}{\sin(\gamma+\alpha)} = \frac{\cos(\gamma+\theta)}{\sin(\alpha+\beta)} = K$$

Find the value of K.

answer. Observe that α, β, γ all satisfy the below equation in x

$$\frac{\cos(x+\theta)}{\sin(S-x)} = k \qquad S = \alpha+\beta+\gamma$$

$$\cos x \times \cos\theta - \sin x \times \sin\theta = k \sin S \cos x - k \sin x \cos S$$

$$\sin x(k\cos S - \sin\theta) = \cos x(k \sin S - \cos\theta) \rightarrow (1)$$

Assume that $(k\cos S - \sin\theta) \neq 0$

$$\tan x = \frac{(k\sin S - \cos\theta)}{(k\cos S - \sin\theta)} = \delta$$

So,

$$\tan\alpha = \tan\beta = \tan\gamma = \delta$$

$$\alpha = n\pi + \beta = m\pi + \gamma \qquad \text{(a contradiction)}$$

So,

$$(k\cos S - \sin\theta) = 0$$

and from (1)

$$(k\sin S - \cos\theta) = 0$$

$$\implies k\cos S = \sin\theta \qquad \text{and} \qquad k\sin S = \cos\theta$$

Squaring and adding, we get

$$k^2(\cos^2 S + \sin^2 S) = (\sin^2\theta + \cos^2\theta)$$

$$k^2 = 1 \implies k = \pm 1$$

□

Problem 1.17. If $x = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\sqrt{3\sqrt{2...}}}}}}$, Find the value of x^2

answer. First, since the pattern is infinite, then we can rewrite x to be $x = \sqrt{3\sqrt{2x}}$. After squaring x, we get $x^2 = 3\sqrt{2x}$. After squaring it again, we get that $x^4 = 18x$. Therefore $x^3 = 18$ and $x = \sqrt[3]{18}$. Substituting it in $x^2 = 3\sqrt{2x}$, we get that $x^2 = \sqrt[3]{18^2}$ □

Problem 1.18. Solve for x

$$\sqrt[4]{1-x^2}+\sqrt[4]{1-x}+\sqrt[4]{1+x}=3$$

answer.

$$((1-x)(1+x))^{\frac{1}{4}}+(1-x)^{\frac{1}{4}}+(1+x)^{\frac{1}{4}}=3$$

Let,

$$a=(1-x)^{\frac{1}{4}}, \qquad b=(1+x)^{\frac{1}{4}}$$

$$ab+a+b+1=3+1$$

$$(1+b)(1+a)=4 \implies a=1 \quad \text{and} \quad b=1$$

Now, we can say that $1+x=1-x$. So, $x=0$. □

Problem 1.19. There are real numbers a, b, c, and d such that -20 is a root of x^3+ax+b and -21 is a root of x^3+cx^2+d. These two polynomials share a complex root $m+\sqrt{n}\cdot i$, where m and n are positive integers and $i=\sqrt{-1}$. Find $m+n$.

answer. Since we know each polynomial has a real root and share the complex root $m+\sqrt{n}i$, the other root must be the complex conjugate which is $m-\sqrt{n}i$.

Applying Vieta's on the equation x^3+ax+b, we find that the sum of the roots is 0. Therefore,

$$-20+(m+\sqrt{n}i)+(m-\sqrt{n}i)=0$$

$$2m=20$$

$$m=10.$$

Applying Vieta's on the equation x^3+cx^2+d, we find that the sum of the product of the roots taken in pairs of 2 is 0. Therefore,

$$(-21)(m+\sqrt{n}i)+(-21)(m-\sqrt{n}i)+(m+\sqrt{n}i)(m-\sqrt{n}i)=0$$

$$-21m-21\sqrt{n}i-21m+21\sqrt{n}i+m^2+n=0.$$

We know that m is 10, so

$$-42(10)+100+n=0$$

$$n=320.$$

Therefore, $m+\sqrt{n}i=10+\sqrt{320}$, so $m+n=330$ □

Problem 1.20. Given that

$$a+\frac{3}{b}=3$$

$$b+\frac{2}{c}=2$$

$$c+\frac{1}{a}=1$$

Find $a+2b+3c$

answer.

$$a+\frac{3}{b}-3=a+\frac{3}{2-\frac{2}{c}}-3=a+\frac{3}{2-\frac{2}{1-\frac{1}{a}}}-3=a+\frac{3}{2-\frac{2a}{a-1}}-3=a-\frac{3(a-1)}{2}-3=\frac{-3-a}{2}$$

So, $a=-3$, $c=\frac{4}{3}$ and $b=\frac{1}{2}$. Substituting these values, we get

$$a+2b+3c=-3+1+4=2$$

$\square$